

Analysis for Malaria Transmission Dynamic between Human and Mosquito Population, Part II: Effective Infection Rate using New Technique

S. Saravana Kumar, L. Maragatham, A. Eswari¹

Department of Mathematics, Sri Ramakrishna Institute of Technology, ¹Department of Physical Sciences and IT, AEC and RI, TNAU, Coimbatore, Tamil Nadu, India

Abstract

Objective: This article presents the seven equation SEIR-SIR model for the dynamics of malaria parasite transmission in both mosquito and human. It defines the presence of area in which the model is epidemiologically feasible. **Material and Methods:** This paper is to find the approximate solution of the above models using q-homotopy analysis method. It is a flexible method that is used to solve a variety of differential equations. **Results:** Numerical simulations are carried out to confirm the analytic results and explore the possible behavior of the formulated model. **Conclusion:** The results of our study are that, Malaria can be controlled by reducing the rate of contact between humans and mosquitoes, the use of active malaria drugs, insecticides and mosquito nets treated with mosquitoes can also help reduce mosquito populations and malaria transmission respectively.

Keywords: Infection rate, malaria transmission, nonlinear model, q-homotopy analysis method, simulation

INTRODUCTION

Malaria, an infectious disease caused by a virus contracted by the bite of a female mosquito called Anopheles mosquito, is one of the most prevalent vector transmitted diseases in the world. Given the decades of global efforts to eliminate the disease and to monitor reappear in areas where preventive measures have been successful and appears in epidemic-free zone.^[1-5] Mathematical models for malaria transmission dynamics are useful to provide a clearer view of the epidemic, to plan for the future, and to suggest effective control steps. Models played a major role in the growth of the disease epidemiology. The study on malaria using mathematical modeling originated from the works of a recent study.^[6] According to Ross, if the mosquito population can be reduced to below a certain threshold, then malaria can be eradicated. MacDonald did some modifications to the model and included superinfection.^[7,8] He has shown that the number of mosquitoes has no impact on the malaria epidemiology in areas with severe transmission.

Based on previous studies add two classes of men in their mathematical models, namely those with low recovery rates

(more infections, vulnerability is greater) and high recovery rate (less infection, less susceptibility).^[8,9] Compartmental models of malaria and differential equations are designed to shape the disease.^[8,10-14] Studies on malaria vectors with bifurcation. Forms of malaria transmission that integrate immunity in humans have been studied.^[11-15] Only epidemiological models were developed for the dissemination of antimalarial resistance.^[15] Anderson and May proposed a malaria model assuming that the immunity obtained in malaria is independent of the period of exposure.^[16] They also examined the different control measures and the role of transmission rates in the prevalence of disease. Ross and Macdonald's works have been expanded further by a research with the common

Address for correspondence: Prof. S. Saravana Kumar, Department of Mathematics, Sri Ramakrishna Institute of Technology, Coimbatore, Tamil Nadu, India. E-mail: saravanatce@gmail.com

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generalized SEIR malaria model, which involves human and mosquito interactions.^[10] Recently, Al-Rahman *et al.* analyzed the mathematical model for malaria transmission using stability analysis.^[17] Currently, previous research has solved a nonlinear model in different mathematical areas using asymptotic method.^[18-21]

Our goal in this work is to investigate and understand the effect of logistic growth model due to the transmission of malaria in a fully homogeneous population and to inspect the influence of the embedded parameters in the dynamic of the malaria disease model. In this paper, we present a model formulation of malaria transmission in part 2, where the general mathematical framework, notation and equations SEIR model-SEI analyzed. In section 3, Application of q-HAM. In Part 4, we perform numerical simulation model with a graphic illustration. Part 5 consists of the results of the discussion and gives a closing speech in Part 6.

MATHEMATICAL ANALYSIS OF SEIR-SEI MODEL

The SEIR-SEI Model for the transmission of malaria between humans and mosquitoes is relevant to this work,^[17] using a seven-dimensional (ODE'S) Ordinary Differential Equations method to model the transmission of plasmodium falciparum malaria between humans and mosquitoes with nonlinear infection rate in the form of saturated incidence rates. This model is formulated for both human population and mosquito population at time t . We divide the human population into (4) four classes: Susceptible S_H , Exposed E_H , Infectious I_H , and Recovery Human R_H , and that of the population the mosquitoes is divided into three classes they are Susceptible S_V , Exposed E_V , and Infectious I_V , respectively. The compartmental model which shows the mode of transmission of malaria between the two interacting populations is depicted in the Figure 1. The above model equations are given by:

$$\frac{dS_H}{dt} = \Lambda_H - \beta_H S_H I_H - \mu_H S_H$$

$$\frac{dE_H}{dt} = \beta_H S_H I_H - (\alpha_{1H} + \mu_H) E_H$$

$$\frac{dI_H}{dt} = \alpha_{1H} E_H - (\alpha_{2H} + \mu_H + \delta) I_H$$

$$\frac{dR_H}{dt} = \alpha_{2H} I_H - \mu_H R_H$$

$$\frac{dS_V}{dt} = \Lambda_V - \beta_V S_V I_V - \mu_V S_V$$

$$\frac{dE_V}{dt} = \beta_V S_V I_V - (\alpha_{1V} + \mu_V) E_V$$

$$\frac{dI_V}{dt} = \alpha_{1V} E_V - \mu_V I_V \tag{1}$$

with

$$S_H(0) > 0, E_H(0) \geq 0, I_H(0) \geq 0, R_H(0) \geq 0, S_V(0) > 0, E_V(0) \geq 0, I_V(0) \geq 0, \text{ and}$$

Where Λ_H denotes recruitment rate for humans, Λ_V is recruitment rate for mosquitoes, α_{1H} is developing rate of exposed (humans) becoming infectious, α_{2H} is recovery rate of humans (removal rate), μ_H is natural death rate for humans, δ is inducing death rate for humans, α_{1V} is developing rate of exposed (mosquitoes) becoming infectious, μ_V is natural death rate for mosquito, and β_H and β_V are infection rate $q_H \times \eta_V$ of humans $q_V \times \eta_V$ and mosquitoes, respectively, where q_H and q_V are probability of transmission of infection from an infectious mosquito to a susceptible human and an infectious human to a susceptible mosquito, respectively. For q-ham solution, we choose the linear operator:^[22,23]

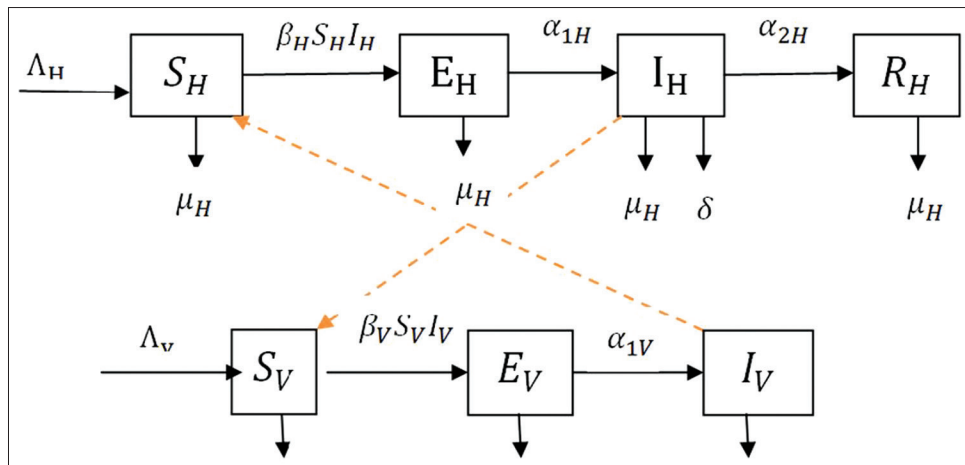


Figure 1: Schematic diagram of malaria transmission between humans and mosquitoes

$$L_{S_H} = \frac{dS_H}{dT} \quad L_{E_H} = \frac{dE_H}{dT} \quad L_{I_H} = \frac{dI_H}{dT} \quad L_{R_H} = \frac{dR_H}{dT}$$

$$L_{S_V} = \frac{dS_V}{dT} \quad L_{E_V} = \frac{dE_V}{dT} \quad L_{I_V} = \frac{dI_V}{dT} \tag{2}$$

Which satisfies the following conditions:

$$L_{S_H}(c_1) = 0, L_{E_H}(c_2) = 0, L_{I_H}(c_3) = 0, L_{R_H}(c_4) = 0,$$

$$L_{S_V}(c_5) = 0, L_{E_V}(c_6) = 0, L_{I_V}(c_7) = 0$$

Where $c_1, c_2, c_3, c_4, c_5, c_6$ and c_7 are integral constants. According to the zeroth-order deformation equations and the m th-order deformation equations with the initial conditions,

$$S_H(0) = S_{H0}, E_H(0) = E_{H0}, I_H(0) = I_{H0}, R_H(0) = R_{H0}, S_V(0) = S_{V0},$$

$$E_V(0) = E_{V0}, I_V(0) = I_{V0} \tag{3}$$

Where

$$\mathfrak{R}_{m,S_H}(t) = S_{H_{m-1}}'(t) + \mu_H S_{H_{m-1}}(t) - \wedge_H$$

$$+ \beta_H \sum_{i=0}^{m-1} S_{H_i}(t) I_{H_{m-1-i}}(t)$$

$$\mathfrak{R}_{m,E_H}(t) = E_{H_{m-1}}'(t) + (\alpha_{1H} + \mu_H) E_{H_{m-1}}(t)$$

$$- \beta_H \sum_{i=0}^{m-1} S_{H_i}(t) I_{H_{m-1-i}}(t)$$

$$\mathfrak{R}_{m,I_H}(t) = I_{H_{m-1}}'(t) + (\alpha_{2H} + \mu_H + \delta) I_{H_{m-1}}(t) - \alpha_{1H} E_{H_{m-1}}(t)$$

$$\mathfrak{R}_{m,R_H}(t) = R_{H_{m-1}}'(t) + \mu_H R_{H_{m-1}}(t) - \alpha_{2H} I_{H_{m-1}}(t)$$

$$\mathfrak{R}_{m,S_V}(t) = S_{V_{m-1}}'(t) + \mu_V S_{V_{m-1}}(t) - \wedge_V + \beta_V \sum_{i=0}^{m-1} S_{V_i}(t) I_{V_{m-1-i}}(t)$$

$$\mathfrak{R}_{m,E_V}(t) = E_{V_{m-1}}'(t) + (\alpha_{1V} + \mu_V) E_{V_{m-1}}(t)$$

$$- \beta_V \sum_{i=0}^{m-1} S_{V_i}(t) I_{V_{m-1-i}}(t)$$

$$\mathfrak{R}_{m,I_V}(t) = I_{V_{m-1}}'(t) + \mu_V I_{V_{m-1}}(t) - \alpha_{1V} E_{V_{m-1}}(t) \tag{4}$$

Now, the solution of system of equations (4)–(10) for $m \geq 1$. We now successively obtain

$$S_{H_1}(t) = (\mu_H S_{H_0} - \wedge_H + \beta_H S_{H_0} I_{H_0}) ht$$

$$S_{H_2}(t) = (-\wedge_H (n+1) + n S_{H_0} (\mu_H + \beta_H I_{H_0})) ht$$

$$+ (\mu_H S_{H_0} - \wedge_H + \beta_H S_{H_0} I_{H_0}) h^2 t$$

$$+ \frac{1}{2} \left((\mu_H + \beta_H I_{H_0}) (\mu_H S_{H_0} - \wedge_H + \beta_H S_{H_0} I_{H_0}) + \beta_H S_{H_0} ((\alpha_{2H} + \mu_H + \delta) I_{H_0} - \alpha_{1H} E_{H_0}) \right) h^2 t^2$$

$$E_{H_1}(t) = ((\alpha_{1H} + \mu_H) E_{H_0} - \beta_H S_{H_0} I_{H_0}) ht$$

$$E_{H_2}(t) = ((\alpha_{1H} + \mu_H) E_{H_0} - \beta_H S_{H_0} I_{H_0}) (nht + h^2 t)$$

$$+ \frac{1}{2} \left((\alpha_{1H} + \mu_H) ((\alpha_{1H} + \mu_H) E_{H_0} - \beta_H S_{H_0} I_{H_0}) - \beta_H S_{H_0} ((\alpha_{2H} + \mu_H + \delta) I_{H_0} - \alpha_{1H} E_{H_0}) - \beta_H I_{H_0} (\mu_H S_{H_0} - \wedge_H + \beta_H S_{H_0} I_{H_0}) \right) h^2 t^2 \tag{6}$$

$$I_{H_1}(t) = ((\alpha_{2H} + \mu_H + \delta) I_{H_0} - \alpha_{1H} E_{H_0}) ht$$

$$I_{H_2}(t) = ((\alpha_{2H} + \mu_H + \delta) I_{H_0} - \alpha_{1H} E_{H_0}) (nht + h^2 t)$$

$$+ \frac{1}{2} \left((\alpha_{2H} + \mu_H + \delta) ((\alpha_{2H} + \mu_H + \delta) I_{H_0} - \alpha_{1H} E_{H_0}) - \alpha_{1H} ((\alpha_{1H} + \mu_H) E_{H_0} - \beta_H S_{H_0} I_{H_0}) \right) h^2 t^2 \tag{7}$$

$$R_{H_1}(t) = (\mu_H R_{H_0} - \alpha_{2H} I_{H_0}) ht$$

$$R_{H_2}(t) = (\mu_H R_{H_0} - \alpha_{2H} I_{H_0}) (nht + h^2 t)$$

$$+ \frac{1}{2} \left(\mu_H (\mu_H R_{H_0} - \alpha_{2H} I_{H_0}) - \alpha_{2H} (\alpha_{2H} + \mu_H + \delta) I_{H_0} - \alpha_{1H} E_{H_0} \right) h^2 t^2 \tag{8}$$

$$S_{V_1}(t) = (\mu_V S_{V_0} - \wedge_V + \beta_V S_{V_0} I_{V_0}) ht$$

$$S_{V_2}(t) = (-\wedge_V (n+1) + n S_{V_0} (\mu_V + \beta_V I_{V_0})) ht +$$

$$(\mu_V S_{V_0} - \wedge_V + \beta_V S_{V_0} I_{V_0}) h^2 t$$

$$+ \frac{1}{2} \left((\mu_V + \beta_V I_{V_0}) (\mu_V S_{V_0} - \wedge_V + \beta_V S_{V_0} I_{V_0}) + \beta_V S_{V_0} (\mu_V I_{V_0} - \alpha_{1V} E_{V_0}) \right) h^2 t^2$$

$$E_{V_1}(t) = ((\alpha_{1V} + \mu_V) E_{V_0} - \beta_V S_{V_0} I_{V_0}) ht$$

$$E_{V_2}(t) = ((\alpha_{1V} + \mu_V) E_{V_0} - \beta_V S_{V_0} I_{V_0}) (nht + h^2 t)$$

$$+ \frac{1}{2} \left((\alpha_{1V} + \mu_V) ((\alpha_{1V} + \mu_V) E_{V_0} - \beta_V S_{V_0} I_{V_0}) - \beta_V S_{V_0} (\mu_V I_{V_0} - \alpha_{1V} E_{V_0}) - \beta_V I_{V_0} (\mu_V S_{V_0} - \wedge_V + \beta_V S_{V_0} I_{V_0}) \right) h^2 t^2 \tag{10}$$

$$I_{V_1}(t) = (\mu_V I_{V_0} - \alpha_{1V} E_{V_0}) ht$$

$$I_{V_2}(t) = (\mu_V I_{V_0} - \alpha_{1V} E_{V_0}) (nht + h^2 t)$$

$$+ \frac{1}{2} \left(\mu_V (\mu_V I_{V_0} - \alpha_{1V} E_{V_0}) - \alpha_{1V} ((\alpha_{1V} + \mu_V) E_{V_0} - \beta_V S_{V_0} I_{V_0}) \right) h^2 t^2 \tag{11}$$

These solutions (5)–(11) are used to construct the analytical solutions given by

$$S_H(t) = S_{H_0}(t) + S_{H_1}(t) \left(\frac{1}{n}\right) + S_{H_2}(t) \left(\frac{1}{n}\right)^2 \tag{12}$$

$$E_H(t) = E_{H_0}(t) + E_{H_1}(t)\left(\frac{1}{n}\right) + E_{H_2}(t)\left(\frac{1}{n}\right)^2 \tag{13}$$

$$I_H(t) = I_{H_0}(t) + I_{H_1}(t)\left(\frac{1}{n}\right) + I_{H_2}(t)\left(\frac{1}{n}\right)^2 \tag{14}$$

$$R_H(t) = R_{H_0}(t) + R_{H_1}(t)\left(\frac{1}{n}\right) + R_{H_2}(t)\left(\frac{1}{n}\right)^2 \tag{15}$$

$$S_V(t) = S_{V_0}(t) + S_{V_1}(t)\left(\frac{1}{n}\right) + S_{V_2}(t)\left(\frac{1}{n}\right)^2 \tag{16}$$

$$E_V(t) = E_{V_0}(t) + E_{V_1}(t)\left(\frac{1}{n}\right) + E_{V_2}(t)\left(\frac{1}{n}\right)^2 \tag{17}$$

$$I_V(t) = I_{V_0}(t) + I_{V_1}(t)\left(\frac{1}{n}\right) + I_{V_2}(t)\left(\frac{1}{n}\right)^2 \tag{18}$$

NUMERICAL SOLUTION

Here, we analyze the behavior of the system numerically using the surveillance estimate of susceptible, exposed, infectious, and recovered mosquito population, as shown in Table 1, and by considering initial conditions, such as.^[16,17]

$$S_H(0) = 0.8, E_H(0) = 0.1, I_H(0) = 0.08, R_H(0) = 0.02, \\ S_V(0) = 0.7, E_V(0) = 0.2, I_V(0) = 0.1$$

The numerical simulations are performed using MATLAB, and the results are shown in Figure 2 to illustrate the behavior of the system with respect to the different values of the model parameters. It can be easily verified that the condition of nonlinear differential equation (1) could be solved using

analytical method and compared with the help of function pdeX in MATLAB. Satisfactory results are observed.

DISCUSSION

The analytical results show that in epidemic situation of model (1), both the human and mosquito populations will exist and get infected. These results are helpful in predicting malaria transmission and how to find an effective way of malaria prevention and control in the model. In Figure 2, it is inferred that the susceptible human and mosquito population increases as time increases and its carrying capacity remains stable, while the exposed and infected human and mosquito population decreases over time. In reality, the plot reveals that the susceptible human populations continue to grow, which means that the disease will remain endemic throughout the population.

CONCLUSIONS

In this paper, we have formulated and analyzed a compartmental malaria transmission model for nonlinear infection rates in human and mosquito populations. The nonlinear model is analyzed using the asymptotic tool q-HAM. The human population has been divided into four compartments: susceptible, exposed, infectious, and recovered, while the mosquito population has been divided into three compartments: susceptible, exposed, and infectious, because the mosquitoes remain life-infectious and have not recovered. The conditions for the system of analytical solution have been determined using q-homotopy analysis method, and these conditions are further justified numerically by considering a given set of parameter values, see Appendix 1: Basic idea of the q-homotopy analysis of

Table 1: Parameters descriptions for SEIR-SEI model (Al-Rahman *et al.*, 2017)

Parameter name	Parameter description	Parameter value
Λ	Recruitment rate of susceptible	1.2
β	Infection rate	0.001
α_1	Developing rate of exposed (humans)	0.1
α_2	Recovery rate of humans (removal rate)	0.0035
μ	Natural death rate	0.03
δ	Inducing death rate	0.089
Λ_H	Recruitment rate for humans	1.2
Λ_V	Recruitment rate for mosquitoes	0.7
α_{1H}	Developing rate of exposed (humans) becoming infectious	0.1
α_{2H}	Recovery rate of humans (removal rate)	0.0035
μ_H	Natural death rate for humans	0.01146
δ	Inducing death rate for humans	0.068
α_{1V}	Developing rate of exposed (mosquitoes) becoming infectious	0.083
μ_V	Natural death rate for mosquito	0.05
q_H	Probability of transmission of infection from an infectious mosquito to a susceptible humans	0.022
q_H	Probability of transmission of infection from an infectious human to a susceptible mosquitoes	0.24
η_V	Mosquitoes' biting rate	0.29
β_H	Infection rate $q_H \times \eta_V$ of humans	0.00638
β_V	Infection rate $q_H \times \eta_V$ mosquitoes	0.0696

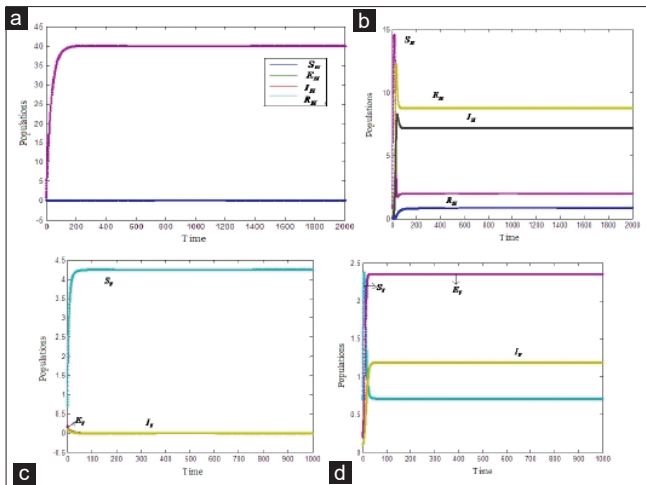


Figure 2(a,b,c,d): Comparison between analytical and Simulation result of the model (1), showing malaria transmission population against time for values of embedded parameters indicated in (Al- Rahman et al., 2017)

the problem. Infection rates are not linear in the model formulated shows the incidence rate is not saturated like ordinary mass action and the common occurrence. We established a region where the model is epidemiologically feasible and mathematically well-posed.

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Conflicts of interest

There are no conflicts of interest.

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APPENDIX

Appendix 1: Basic idea of the q-homotopy analysis of the problem

Considering the following system of differential equation of the form

$$N_i[x_i(t)] - f_i(t) = 0, \quad i = 1, 2, 3, \dots, n \tag{A1}$$

Where $N_i = (i = 1, 2, 3, \dots, n)$ are nonlinear operators, t is independent variable, $f_i(t) (i = 1, 2, 3, \dots, n)$ are

Known functions and $x_i(t) (i = 1, 2, 3, \dots, n)$ are an unknown function. Note that, not necessary in the nonlinear operator N_i contains nonlinear term. Let us construct the so-called zeroth-order deformation equation

$$(1 - nq)L[\varphi_i(t; q) - x_{i,0}(t)] = qhH_i(t)[N_i(\varphi_i(t; q)) - f_i(t)] \tag{A2}$$

Where $n \geq 1$ $q \in \left[0, \frac{1}{n}\right]$ denotes the so-called embedding operator, L is an auxiliary linear operator with the property $L[f] = 0$ when $f = 0, h \neq 0$ is an auxiliary parameter, $x_{i,0}(t)$ are initial guesses of $x_i(t)$ and $H_i(t)$ denotes a non-zero auxiliary function.

It is obvious that when $q = 0$ and $q = \frac{1}{n}$ equation (A2) becomes

$$\varphi_i(t; 0) = x_{i,0}(t) \text{ and } \varphi_i\left(t, \frac{1}{n}\right) = x_i(t) \tag{A3}$$

respectively. Thus, as q increases from 0 $\frac{1}{n}$, to the solution $\varphi_i(t; q)$ varies from the initial guess $x_{i,0}(t)$ to the solution $x_i(t)$. Having the freedom to choose $x_{i,0}(t), L, h, H_i(t)$, we can assume that all of them can be properly chosen so that the solution of $\varphi_i(t; q)$ equation (A2) exists for $q \in \left[0, \frac{1}{n}\right]$.

Expanding $\varphi_i(t; q)$ in Taylor series with respect to q , one has

$$\varphi_i(t; q) = x_{i,0}(t) + \sum_{m=1}^{\infty} x_{i,m}(t)q^m \tag{A4}$$

Where

$$x_{i,m}(t) = \frac{1}{m!} \frac{\partial^m \varphi_i(t; q)}{\partial q^m} \Big|_{q=0} \tag{A5}$$

Assume that $h, H_i(t), x_{i,0}(t), L$ are properly chosen such that the series (A4) converges at $q = \frac{1}{n}$ and:

$$x_i(t; q) = \varphi_i\left(t; \frac{1}{n}\right) = x_{i,0}(t) + \sum_{m=1}^{\infty} x_{i,m}(t) \left(\frac{1}{n}\right)^m \tag{A6}$$

Define the vector $\overline{x_{i,n}}(t; q) = \{x_{i,0}(t), x_{i,1}(t), \dots, x_{i,n}(t)\}$,

Differentiating equation (A2) m times with respect to the embedding operator q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called m th-order deformation equation

$$L[x_{i,m}(t) - \mathcal{L}_m x_{i,m-1}(t)] = hH_i(t)R_{i,m}(\overline{x_{i,m-1}}), \tag{A7}$$

Where

$$R_{i,m}(\vec{x}_{i,m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\varphi_i(t; q)]}{\partial q^{m-1}} \Big|_{q=0}, \quad (A8)$$

and

$$\zeta_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (A9)$$

It should be emphasized that is $x_{i,m}(t)$ $m \geq 1$ governed by the linear equation (A7) with the linear boundary conditions that come from the original problem. Due to the existence of the factor $\left(\frac{1}{n}\right)^m$, more chances for convergence may occur or even much faster convergence can be obtained better than standard HAM. It should be noted that in the case of $n = 1$ in equation (A2), standard HAM can be reached.